Closing Thu: TN 5

Final Sat, June 2nd,
5:00-7:50pm,
KANE 130

Seating

CA, CB (Collin) – Balcony
All other sections – Main Floor

Entry Task:
Give a Taylor series answer for

$$\int_0^1 e^{-t^2}\,dt$$

RECALL

$$e^{z} = \sum_{k=0}^{\infty} \frac{1}{k!} z^k \quad \text{for all } z.$$

THUS,

$$e^{-t^2} = \sum_{k=0}^{\infty} \frac{1}{k!}(-t^2)^k = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} t^{2k}$$

SO

$$\int e^{-t^2}\,dt = C + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{1}{(2k+1)} t^{2k+1}$$

$$\Longrightarrow \int_0^1 e^{-t^2}\,dt = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(2k+1)} t^{2k+1}\Big|_0^1$$

$$= \boxed{\sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(2k+1)}}$$

HERE ARE THE FIRST FEW TERMS:

$$\int_0^1 e^{-t^2}\,dt = \frac{1}{0!\,(1)} - \frac{1}{1!\,(3)} + \frac{1}{2!\,(5)} - \frac{1}{3!\,(7)} + \frac{1}{4!\,(9)} - \frac{1}{5!\,(11)} + \cdots$$

$0.\overline{6}$

$0.7\overline{6}$

0.74286

0.747487

0.746729

OUT TO $k=10$ GIVES

$0.\underline{74682413}38$

MATCHES ACTUAL ANSWER

Example (*from HW*)

Write down the Taylor series for $\sin(t)$ based at 0. Then use it to give the Taylor series for

$$A(x) = \int_0^x \frac{\sin(t)}{t}\,dt$$

$$\sin(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} t^{2k+1} = t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \cdots$$

$$\frac{\sin(t)}{t} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} t^{2k} = 1 - \frac{1}{3!}t^2 + \frac{1}{5!}t^4 - \cdots$$

$$\Rightarrow \int_0^x \frac{\sin(t)}{t}\,dt = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \frac{1}{(2k+1)} t^{2k+1}\Big|_0^x = x - \frac{1}{3!}\frac{1}{3}x^3 + \frac{1}{5!}\frac{1}{5}x^5 - \frac{1}{7!}\frac{1}{7}x^7 + \cdots$$

$$= \boxed{\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!\,(2k+1)} x^{2k+1}}$$

Math 126 Final

Saturday, June 2

5:00-7:50pm in Kane 130

CA/CB (Collin) in balcony

All other section on main floor.

1. Bring some form of photo ID. We will check ID's.

2. Final grades will be posted by Friday of next week.

3. Allowed:
   (a) One 8.5 by 11 inch sheet of handwritten notes (front and back)
   (b) Ti-30x IIS calculator (this model only)

4. ***Coverage***

Eight pages of questions.

Exam is comprehensive (covers everything).

Quick Review:

Ch. 12: 3D Basics (vector facts, lines, planes, basic surfaces, ...)

Ch 13: 3D Curves (accel/vel/position, tangent vector, unit tangent, tangent line, normal vector, curvature, ...)

Ch 14/15: 3D Surfaces (traces, partial deriv, max/min, double integrals, ...)

TN: Taylor Polynomials and Series (use deriv. to find Taylor Poly., error bounds, Taylor series patterns, ...)

A Recent Final Question on Taylor Polynomials and Series

Winter 2018 / Problem 7

Let $f(x) = x^2 \sin(x^3) + \frac{1}{8-x^3}$.

(a) Find the 6th Taylor polynomial based at 0. GO TO x^6!

(b) Give the open interval of convergence of the Taylor series for f(x) based at 0.

(a) $x^2 \sin(x^3)$

$$= x^2 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}(x^3)^{2k+1} = x^2\left((x^3) - \frac{1}{3!}(x^3)^3 + \frac{1}{5!}(x^3)^5 - \cdots\right)$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{6k+5} = \underbrace{x^5}_{\text{STOP}} - \frac{1}{3!}x^{11} + \frac{1}{5!}x^{17} \cdots$$

$\boxed{\text{FOR ALL } x}$

$$\frac{1}{8-x^3} = \frac{1}{8}\left(\frac{1}{1-\frac{1}{8}x^3}\right)$$

$\boxed{\text{FOR } -1 < \frac{1}{8}x^3 < 1}$

$$= \frac{1}{8}\sum_{k=0}^{\infty}\left(\frac{1}{8}x^3\right)^k$$

$$= \frac{1}{8}\sum_{k=0}^{\infty}\frac{1}{8^k}x^{3k} = \frac{1}{8}\left(\underbrace{1 + \frac{1}{8}x^3 + \frac{1}{64}x^6}_{\text{STOP!}} + \cdots\right)$$

THUS

$$x^2\sin(x^3) + \frac{1}{8-x^3} \approx x^5 + \frac{1}{8} + \frac{1}{64}x^3 + \frac{1}{512}x^6$$

$$\boxed{T_6(x) = \frac{1}{8} + \frac{1}{64}x^3 + x^5 + \frac{1}{512}x^6}$$

(b) $-1 < \frac{1}{8}x^3 < 1$

$\Rightarrow -8 < x^3 < 8$

$\Rightarrow \boxed{-2 < x < 2}$

Winter 2018 / Problem 8

Let $g(x) = \sqrt{3+x^2}$.

(a) Find the 1[st] Taylor polynomial based at 1.

(b) Give a bound on the error over the interval [0.5, 1.5].

(a) $g(x) = (3+x^2)^{1/2} \Rightarrow g(1) = 2$

$g'(x) = \frac{1}{2}(3+x^2)^{-1/2} \cdot 2x = \frac{x}{\sqrt{3+x^2}} \Rightarrow g'(1) = \frac{1}{2}$

$$\boxed{T_1(x) = 2 + \frac{1}{2}(x-1)}$$

(b) $g''(x) = (3+x^2)^{-1/2} \cdot (1) - \frac{1}{2}(3+x^2)^{-3/2} 2x \cdot x$

$$= \frac{1}{\sqrt{3+x^2}} - \frac{x^2}{(3+x^2)^{3/2}}$$

COMMON DENOMINATOR

$$\frac{3+x^2}{(3+x^2)^{3/2}} - \frac{x^2}{(3+x^2)^{3/2}}$$

$$g''(x) = \frac{3}{(3+x^2)^{3/2}}$$

$$|g''(x)| = \frac{3}{(3+x^2)^{3/2}} \leq \frac{3}{(3+0.5^2)^{3/2}}$$

[0.5, 1.5]

$$M = \frac{3}{(3.25)^{3/2}} \simeq 0.51203$$

$$\text{Error} \leq \frac{M}{2!}|x-1|^2$$

$$\leq \frac{0.51203}{2}|0.5-1|^2$$

$$= \boxed{0.064}$$